Quiz 2 Solutions

1). $f(x) = \dfrac{\sin x}{\sqrt{x}}$ is uniformly continuous on $(0,\infty)$

Consider $f(x)$ in $(0,1]$

Define $\tilde{f}(x) = \begin{cases} f(x) & x \neq 0. \\ 0 & x = 0. \end{cases}$ $\tilde{f}(x)$ is continuous on $[0,1]$. as. $\lim\limits_{x\to 0} \dfrac{\sin x}{\sqrt{x}} = 0$.

$\tilde{f}$ is uniformly continuous on $[0,1]$.

$\Rightarrow f$ is uniformly continuous on $(0,1]$ 2

Now consider $f(x)$ in $[1,\infty)$.

$$f'(x) = \frac{(\cos x)\sqrt{x} - \sin x \cdot \frac{1}{2\sqrt{x}}}{x}$$

$$|f'(x)| = \left|\frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x^{3/2}}\right| \leq \left|\frac{\cos x}{x}\right| + \left|\frac{\sin x}{2\cdot x^{3/2}}\right| \leq \left(\left|\frac{1}{x}\right| + \frac{1}{2x^{3/2}}\right).$$

$x > 1 \Rightarrow \frac{1}{x} < 1 \Rightarrow$. $|f'(x)| \leq 1 + \frac{1}{2} = \frac{3}{2}$

$|f'(x)|$ is bounded $\Rightarrow$. f is uniformly continuous on $[1,\infty)$. 2

$\Rightarrow f$ is uniformly continuous on $(0,\infty)$.

2) a). $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$. is convergent for $|x| < R$

f is differentiable & $f'(x) = \sum\limits_{n=0}^{\infty} a_n n x^{n-1}$ for $|x| < R$. 2

If f is differentiable then f is continuous. 1

$\Rightarrow$. $|f(x+h) - f(x)| \to 0$ as $h \to 0$. 1

b). Radius of convergence of $\sum \frac{a_n}{8^n} x^n$.

$$a_n = \begin{cases} 3^n & n \text{ is odd} \\ 3^{-n} & n \text{ is even.} \end{cases}$$

$$\sum \frac{a_n}{8^n} x^n = \sum b_n x^n \quad \text{where} \quad b_n = \begin{cases} 3^n/8^n. & n \text{ is odd.} \\ \frac{1}{3^n 8^n} & n \text{ is even.} \end{cases}$$ (1)

$$\beta = \limsup \sqrt[n]{|a_n|} = \frac{3}{8}.$$ (2)

$\Rightarrow$ $R = \frac{1}{\beta} = \frac{8}{3}$ is the radius of convergence. (1)

3) Any counterexample works. (3).